Waves and Dispersion

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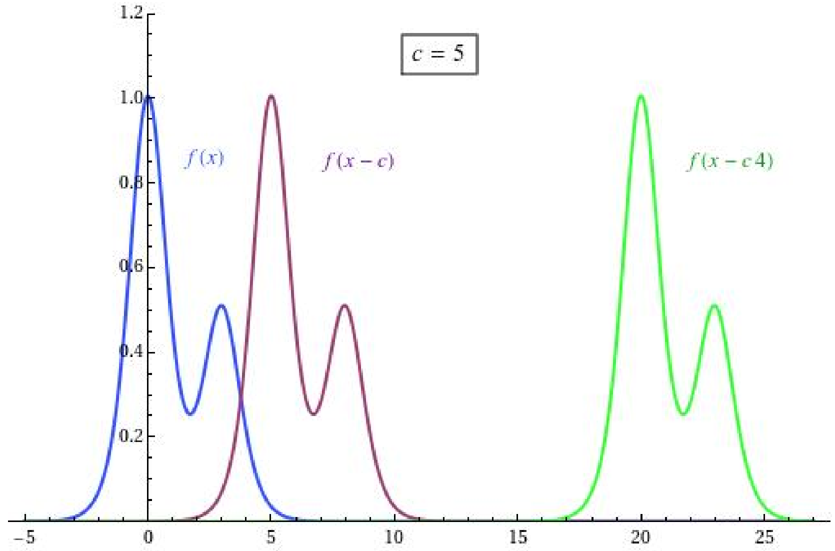
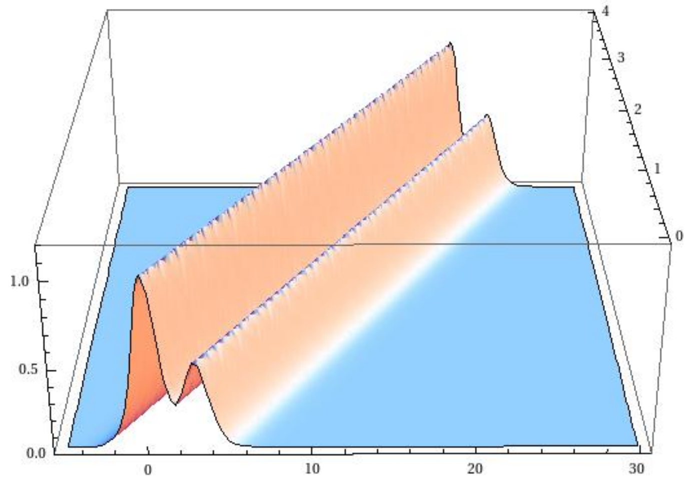
Mathematical Definition of a Wave

The prototype: a traveling wave.

$$\boxed{u(x,t) = f(x - ct), \qquad c \in \mathbb{R}}$$





The Wave Equation in 1D

D'Alembert (1747) derived the one dimensional wave equation to describe the vibration of a string.

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad\qquad \left(c^2 = \frac{T}{\mu}\right)}$$

Performing the change of variables $\xi = x - ct$, $\eta = x + ct$ the equation becomes,

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

whose general solution is simply

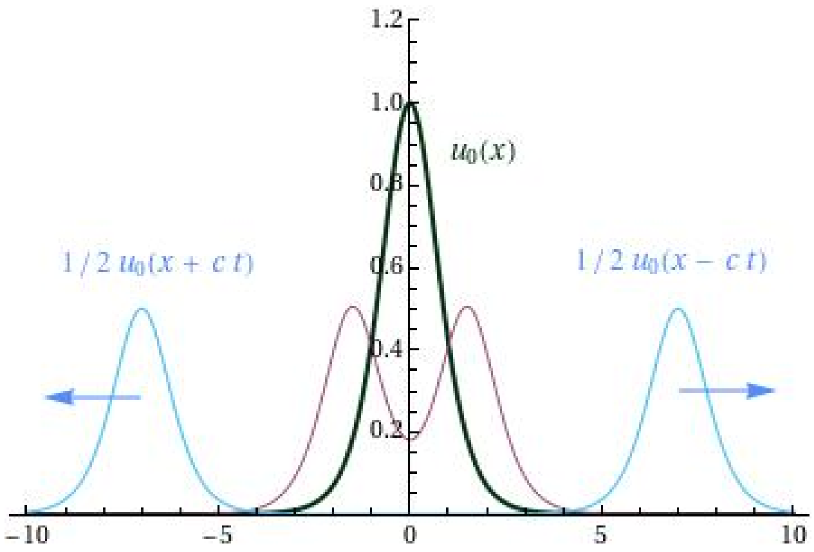
$$\boxed{u = f(\xi) + g(\eta) = f(x - ct) + g(x + ct)}$$

In particular, if one considers the initial value problem for the wave equation, with initial data at $t = 0$

$$u(x,0) = u_0(x) \qquad \frac{\partial u}{\partial t}(x,0) = 0,$$

then the explicit solution becomes

$$\boxed{u(x,t) = \frac{1}{2}\big(u_0(x-ct) + u_0(x+ct)\big)}$$



The Fourier Transform

Under general conditions, a function $f : \mathbb{R} \mapsto \mathbb{R}$ can be written as

$$\boxed{f(x) = \int_{\mathbb{R}} a(\xi) e^{2\pi i \xi x} d\xi}$$

where $e^{2\pi i \xi x} = \cos(2\pi\xi x) + i \sin(2\pi\xi x)$ so that

$$f(x \pm ct) = \int_{\mathbb{R}} a(\xi) e^{2\pi i(\xi x \pm \xi c t)} d\xi.$$

A general travelling wave can thus be considered as a (continuous) superposition of sinusoidal waves of different frequencies and amplitudes

$$\boxed{a(\xi)\, e^{2\pi i(\xi x + \tau t)}, \qquad \text{with} \qquad \tau = \pm \xi c}$$

Plane Waves

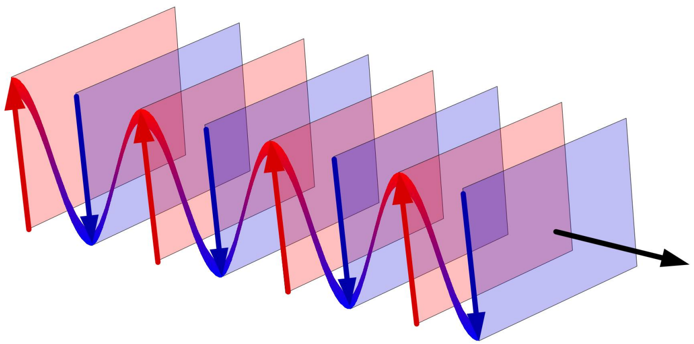
The simplest generalization of the traveling wave in higher dimensions.

If $\vec{\mathbf{e}} \in \mathbb{R}^n$, $\|\vec{\mathbf{e}}\| = 1$ is a unit vector defining a direction in space

and $f : \mathbb{R} \to \mathbb{R}, \mathbb{C}$ is function of just one variable, then

$$\boxed{u(\vec{x}, t) = f(\vec{\mathbf{e}} \cdot \vec{x} \pm ct) \qquad (\vec{x}, t) \in \mathbb{R}^n \times \mathbb{R}}$$

is a plane traveling wave along the direction defined by $\vec{\mathbf{e}}$ with velocity c.



In the case of a sinusoidal function, for example $f(x) = \cos(2\pi\xi x)$

$$u(\vec{x}, t) = \cos\left(2\pi\xi(\vec{\mathbf{e}} \cdot \vec{x} \pm ct)\right) = \cos\left(2\pi(\vec{\xi} \cdot \vec{x} \pm \tau t)\right),$$

with spatial frequency $\vec{\xi} = \xi\vec{\mathbf{e}}$ and time frequency $\tau = \xi c$.

Generally, as before, a space-time function can then be considered as a (continuous) superposition of these sinusoidal plane waves with different space and time frequencies

$$u(\vec{x},t) = \int a(\vec{\xi},\tau) e^{2\pi i(\vec{\xi}\cdot\vec{x}+\tau t)} d\vec{\xi} d\tau$$

The basic building blocks of general waves are then the individual exponential plane waves

$$\boxed{e^{2\pi i(\vec{\xi}\cdot\vec{x}+\tau t)} = e^{2\pi i|\vec{\xi}|(\vec{\mathbf{e}}_{\vec{\xi}}\cdot\vec{x}+\frac{\tau}{|\vec{\xi}|}t)}}$$

where $\vec{\xi} = |\vec{\xi}|\vec{\mathbf{e}}_{\vec{\xi}}$ space frequency and $\tau \in \mathbb{R}$ time frequency.
The speed of propagation of the plane wave along the direction defined by $\vec{\xi}$ is given by

$$\boxed{\vec{v} = -\frac{\tau}{|\vec{\xi}|}\vec{\mathbf{e}}_{\vec{\xi}}}$$

and is called the phase velocity.

Dispersion Relation

In mathematical models of physics described by partial differential equations the space and time frequency of basic plane waves are not free but constrained by the equation.

Wave Equation ($\vec{x} \in \mathbb{R}^n$):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta_x u \quad \Rightarrow \quad \boxed{\tau = \pm\, c\,|\vec{\xi}|, \quad \vec{v} = \pm\, c\,\vec{\mathbf{e}}_{\vec{\xi}}}$$

Free Schrödinger Equation ($\vec{x} \in \mathbb{R}^n$):

$$i\frac{\partial u}{\partial t} = -\Delta_x u \quad \Rightarrow \quad \boxed{\tau = -2\pi|\vec{\xi}|^2, \quad \vec{v} = 2\pi|\vec{\xi}|\vec{\mathbf{e}}_{\vec{\xi}}}$$

Airy Equation ($x \in \mathbb{R}$):

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \quad \Rightarrow \quad \boxed{\tau = 4\pi^2\xi^3, \quad v = -4\pi^2\xi^3/|\xi|}$$

Dispersive Equations

A linear partial differential equation is called dispersive if the phase velocity of plane waves is real and not constant.

Heat Equation ($\vec{x} \in \mathbb{R}^n$):

$$\frac{\partial u}{\partial t} = \Delta_x u \quad \Rightarrow \quad \boxed{\tau = 2\pi i|\vec{\xi}|^2, \quad \vec{v} = -2\pi i|\vec{\xi}|\vec{\mathbf{e}}_{\vec{\xi}}}$$

The heat equation is NOT dispersive.

Nonlinear Dispersive Equations and Solitary Waves

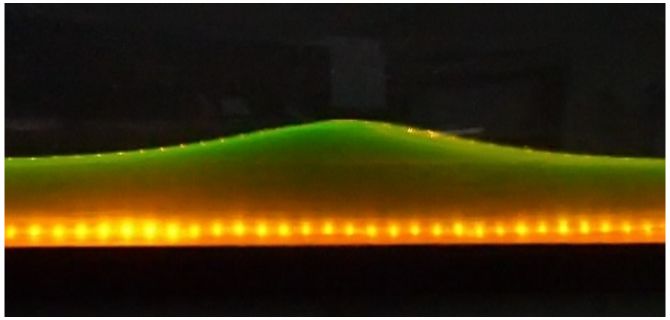
Nonlinear Schrödinger Equation (NLS):

$$i\frac{\partial u}{\partial t} = -\Delta_x u + |u|^p u$$

Korteweg-DeVries (KdV):

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = u\frac{\partial u}{\partial x}$$

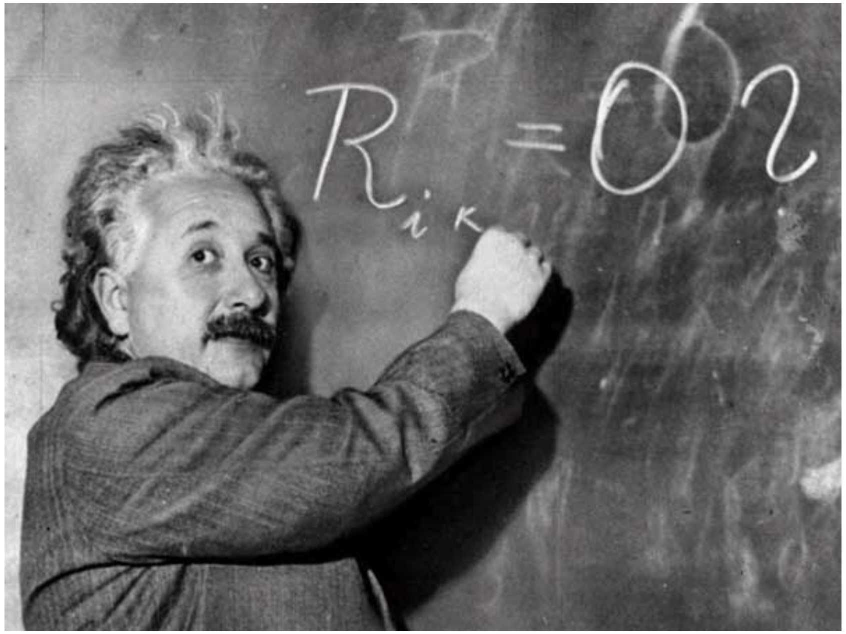
Solitons



Einstein Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

In vaccuum and with cosmological constant $\Lambda = 0$,



That's All Folks